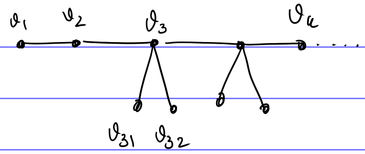

$\{0, 1, 2, 3, 4\}$

Let m be even, then the # of vertices $n = m+1$ is odd.

Let the spine vertices be labeled $\{v_1, v_2, \dots\}$ and the neighbors off the spine of v_k be labeled $\{v_{kj}\}_{j=1}^{N_k}$. The labels are $\{0, \dots, m\}$

Algorithm: We simply alternate between choosing the

A: SMALLEST AVAILABLE LABEL and the

B: LARGEST " "

But this depends on the situation. We proceed inductively.

Suppose we have labeled upto v_k so far.

If v_k labeled using A, then $\{v_{kj}\}$ and v_{k+1} must receive labels using B.

Claim: This produces a graceful labeling.

At $k=0$, the first vertex is labeled 0 (wlog). Then the next one is m. At stage k, we have just labeled vertex a_k using A, and the difference $m-k$ has appeared

We need to consider the labels of a_k and a_{k+1}. Suppose a_k is on the spine and labeled with A. Then $\exists$ some a_j on the spine, st a_k is its neighbor and by construction a_j must have label in B, and the difference $f(a_j) - f(a_k) = m-k$ has just appeared. Then a_{k+1} must have label in B, and $f(a_{k+1}) = f(a_j) - 1$

Now the differen $f(a_{k+1}) - f(a_k) = f(a_j) - f(a_k) - 1 = m-k-1$ appears.

The case when a_k is not on the spine can be argued similarly. We are done by induction.

α = min over all labels produced using B.